Calculating Eigenvalues of 2×2 Matrices

Recall: Let A be a $n \times n$ matrix. We call a vector $\boldsymbol{x}$ an *eigenvector* of A with corresponding *eigenvalue* λ (a scalar) if

$$A\boldsymbol{x} = \lambda\boldsymbol{x}, \qquad \boldsymbol{x} \neq \mathbf{0} \tag{1}$$

Theorem 1: Let A be a $n \times n$ matrix. Then the scalar λ is an eigenvalue of A if and only if there exists a vector $\boldsymbol{x}$ such that

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}, \qquad \boldsymbol{x} \neq \mathbf{0} \tag{2}$$

Proof:

$(A - \lambda I)\vec{x} = \vec{0}, \quad \vec{x} \neq \vec{0} \iff$

$A\vec{x} - \lambda I\vec{x} = \vec{0}, \quad \vec{x} \neq 0 \iff$

$A\vec{x} = \lambda\vec{x} = \vec{0}, \quad \vec{x} \neq 0 \iff$

λ is an eigenvalue of A.

Corollary 1: Let A be a $n \times n$ matrix. The scalar λ is an eigenvalue of A if and only if

$$\text{nullity}(A - \lambda I) > 0 \tag{3}$$

or equivalently

$$\text{The matrix } A - \lambda I \text{ is } \textit{not} \text{ invertible.} \tag{4}$$

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

Recall: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if and only if $\det(A) \neq 0$.

Corollary 2: Let A be a 2×2 matrix. Then a scalar λ is an eigenvalue of A if and only

$$\det(A - \lambda I) = 0 \tag{5}$$

Example: Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right)$

$= (1-\lambda)^2 - 1$

$= \lambda^2 - 2\lambda + 1 - 1 \Rightarrow \lambda^2 - 2\lambda = \lambda(\lambda - 2)$

The eigenvalues of A are

$\lambda_1 = 0$ and $\lambda_2 = 2$.

Example: Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.

$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix}\right) =$

$(1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4$

$= (\lambda - 2)^2$

The eigenvalue of A is $\lambda = 2$.